Survey of Homological Mirror Symmetry Conjecture

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Abstract

This survey is a combination of the understanding and calculation of the materials coming from several papers [1, 2, 4, 8] in HMS and . This survey reorganizes several crucial results [1, 2] in this field and illustrates the theorem with clear calculation on the examples if possible. We hope that this approach can make the abstract argument in the proof explicit and concrete.

1 Introduction

This is a survey on the topic of homological mirror symmetry(HMS), which is now a rapidly-developing, heated field in both mathematics and physics. The mirror symmetry first shows up in physics and is concerned with a duality between the two fundamental physical objects called brane and string. Then, after the seminal work of Yau [19, 20], mathematicians come in to rigorize the argument which has been conjectured by physicians. It's interesting to find that mathematics always serves as the role to build the fundamental block in the natural phenomenon, which is also the reason that I choose HMS as my research topic in the undergraduate study. Personally, the beauty of HMS lies in the fact that it combines a great range of knowledge together like both mathematics and physics, both geometry, algebra and analysis, both symplectic and algebraic geometry. All of this intrigues me and motivates me to seek the deep connection between all the things behind HMS.

Due to the impetus from great many masters like Kontsevich [10], Seidel [13, 15], Zaslow, Nadler, mirror symmetry has developed a lot during the past two or three decades. Meanwhile, different schools provide disparate perspectives to view such an immense object. For example, there're approaches via the SYZ conjecture, led by the Zaslow et al. [3, 16], and HMS formulation first founded by Kontsevich [10], now led by the school of Seidel, Abouzaid. This survey mainly focuses on the approach via HMS due to its use of more fancy and a modern language like derived category and A_∞-category. The author believes that the prevalence of this language from abstract category theory will help us understand deeper math hidden in the mirror symmetry.

However, a crucial point has to be kept in mind. We approach such problems through a purely abstract perspective and language with great many complicated algebraic and categorical structures defined in it. It's important to understand it's interpretation in the geometric context, which is undoubtedly the origin of such delicate definition. In most cases, the geometry shade light to help us quickly catch the philosophy underlying everything. And usually, the geometry is where this philosophy lies in.

Here we first use a diagram drawn by D.Nadler in his lecture video of Arboreal Singularity [12] [18] [11] to finish the introduction.

Motivation	Topology	Algebra	Analysis	Combinatorials
X manifold, $H^*(X)$, 1d TFT	Smg cochain	De Rham cochain	Morse/Hodge theory	Cellular cochain
M sympl. manifold 2d TFT, Γ Lagrangian skeleton,$\mathscr{C}_\Gamma(M)$	Microlocal sheaves	Modules over deformation quantization	Fukaya category	?

Table 1: A brief table for mirror symmetry

2 On Floer Homology

The *Floer cohomology* is introduced by Floer in 1989 as a method to compute the algebraic invariant of a symplectic manifold. The idea of *Floer cohomology* benefited a lot from the idea of *Morse homology*, in which the topology and the structure of the manifold are studied via the function on this manifold. Without much rigorous, we can say that the *Floer cohomology* is the symplectic counterpart of the *Morse homology* on real manifold. In the following we will provide a crash introduction to the *Floer cohomology* which emphasize the connection with *Morse homology* and is illustrated by the concrete example of $T^*S^1 \simeq S^1 \times \mathbb{R}$ and $T^*\mathbb{R} \simeq \mathbb{R}^2$. We suppose the readers have some elementary knowledge about symplectic geometry.

2.1 Inspiration of Floer cohomology

We review several essential conclusion in *Floer cohomology*, on the goal to provide the essential background of the development of the *Floer cohomology*. The *Floer cohomology* is partly inspired by the well-known conjecture by Arnold, which conjecture the relationship between the intersection number of the Lagrangian submanifolds and some cohomology invariants. And as preparation, we state the main setting of the following result. Specifically, let (M, ω) be a symplectic manifold with a Lagrangian submanifold L such that $\omega_{|L} = 0$ and Ψ a Hamiltonian diffeomorphism.

Theorem 1. *(Floer [6, 7]).Assume that the symplectic area of any topological disc in M with boundary in L vanishes. Assume moreover that $\Psi(L)$ and L intersect transversely. Then the number of intersections points of L and $\Psi(L)$ satisfies the lower bound $|\Psi(L) \cap L| \geq \sum_i \dim H^i(L, \mathbb{Z}_2)$.*

Floer's approach is to associate to the pair of Lagrangians $(L_0, L_1) = (L, \Psi(L))$ a chain complex $CF(L_0, L_1)$, freely generated by the intersection points of L_0 and L_1, equipped with a differential $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$, with the following properties:
(1) $\partial^2 = 0$, so the *Floer cohomology* $HF(L_0, L_1) = Ker\partial / Im\partial$ is well-defined;
(2) if L_0 and L_1 are Hamiltonian isotopic then $HF(L, L_0) \simeq HF(L, L_1)$;
(3) if L_1 is Hamiltonian isotopy to L_0, then $HF(L_0, L_1) \simeq H^*(L_0)$(with suitable coefficients).
Thus the theorem follows easily from the last property and comparison of complex.

2.2 Connection with Morse Homology

While the *Morse homology* serves as the origin of the *Floer cohomology*, consider the following context, we view $L_0 = \mathbb{R}$ as a Lagrangian submanifold(embedded as the zero section) of the symplectic manifold $M = T^*\mathbb{R} \simeq \mathbb{R}^2$ with canonical symplectic form $\omega = dx \wedge dy$ and projection π:

$$\pi : T^*\mathbb{R} \simeq \mathbb{R}^2 \to \mathbb{R} \qquad (x, y) \mapsto x.$$

Then as in *Morse homology*, we consider a morse function f on $\mathbb{R}$, for example, we take:

$$f(x) = \frac{x^2 - 1}{2}.$$

Then we know that $x = 0$ is the unique critical point of the morse function with index 1. In symplectic geometry, we also consider the Hamiltonian isotopy on $T^*\mathbb{R}$ induced by the function ϵf with $(\epsilon \ll 1)$:

$$T^*\mathbb{R} \xrightarrow{\pi} \mathbb{R} \xrightarrow{f} \mathbb{R}.$$

Note that the first $\mathbb{R}$ refers to Lagrangian submanifold and the second is the value space. The Hamiltonian vector field X_t corresponds to this isotopy reads:

$$\omega(X_t, \cdot) = df(\cdot) \Rightarrow X_t = x\frac{\partial}{\partial y}. \tag{1}$$

Thus, suppose the Hamiltonian isotopy transfer the Lagrangian submanifold L_0 to L_1, we can draw the picture as below:

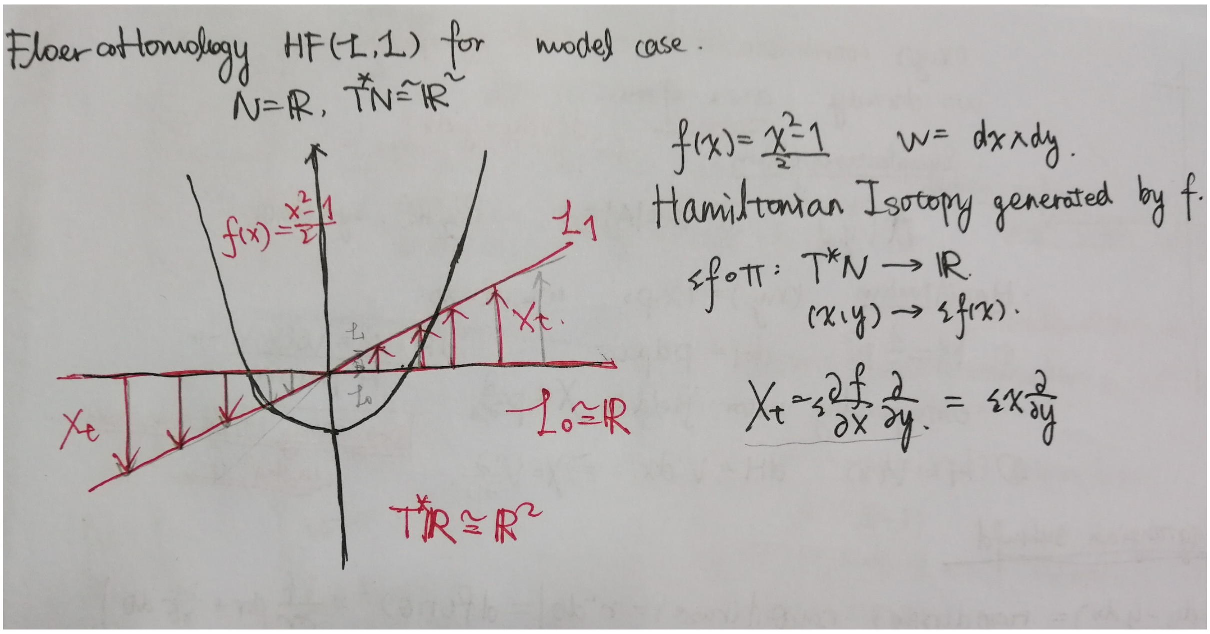

Figure 1: Hamiltonian Isotopy

It can be seen that the critic point of morse function f, i.e. $(0,0)$ coincides with that the intersection point of the two Hamiltonian isotopic Lagrangian. And the Hamiltonian vector field is the gradient field of the given morse function. Thus from the following illustration, we know that while in *Morse homology* we study the critic point of the morse function, in *Floer cohomology* we study the intersection point of the Lagrangian. And this is the inspiration of the Floer's idea.

2.2.1 The Connection Between Gradient Flow in Morse Homology and Pseudo-holomorphic Curves in Floer Homology

Also, in the context of *Morse homology*, we define the differential by counting the gradient flow that connects two critic points p and q, that's:

$$\begin{gathered} \gamma : \mathbb{R} \to L_0 \qquad s.t. \\ \dot{\gamma}(s) = -\epsilon \nabla f(\gamma(s)) \qquad \lim_{s \to -\infty} \gamma(s) = p, \qquad \lim_{s \to \infty} \gamma(s) = q. \end{gathered} \tag{2}$$

The integral coefficients of q in ∂p are exactly the counting number of the following gradient flow in some sense. While in the symplectic context, such counting problem transfers to the counting problem of the pseudo-holomorphic curves, that's, holomorphic curve w.r.t. the given almost complex structure J compatible with ω on M and the ordinary complex structure on the infinite strip $\mathbb{R} \times [0,1]$ that also satisfies some boundary condition as well as some energy estimate which aims to ensure the compactness of the moduli space. Suppose we have two Lagrangian L_0 and L_1 that intersect transversally at two points p and q. The condition that pseudo holomorphic curves have to satisfies reads:

$$\begin{aligned} & u : \mathbb{R} \times [0,1] \to M, \\ & u(s,0) \in L_0, \qquad u(s,1) \in L_1, \\ & \lim_{s \to -\infty} u(s,0) = p, \qquad \lim_{s \to \infty} u(s,1) = q, \\ & \frac{\partial u}{\partial s} + J(u(s,t)) \frac{\partial u}{\partial t} = 0, \\ & E(u) = \int u^* \omega = \int \int \left| \frac{\partial u}{\partial s} \right| dsdt < \infty. \end{aligned} \tag{3}$$

At a first glance, this equation 3 satisfied by a pseudo-holomorphic curve may have nothing to do with that 2 in ordinary *Morse homology* except the boundary condition satisfied by them. If we look closer at the Cauchy-Riemann equation $\bar{\partial}_J u = 0$, the term $\frac{\partial u}{\partial s}$ corresponds to the horizontal derivative of the curves while $\frac{\partial u}{\partial t}$ corresponds to the vertical derivative, since we are considering the infinitesimal Hamiltonian isotopy($\epsilon \ll 1$), and the path $u(s, \cdot)$ connects $u(s, 0) \in L_0$ and $u(s, 1) \in L_1$. Thus approximately, we have the vertical derivative of curves coincides with that of the Hamiltonian X_t:

$$\frac{\partial u}{\partial t} = X_t = -\epsilon \nabla f(u(s, 0)). \tag{4}$$

And the almost complex structure J rotate this vector by $\frac{\pi}{2}$. And, plug this into the Cauchy-Riemann equation we conclude that:

$$\frac{\partial u}{\partial s} = -\epsilon \nabla f(u(s, 0)). \tag{5}$$

Thus a key observation is that, if we restrict the curves on $\mathbb{R} \times \{0\}$, we get a gradient flow in the context of *Morse homology*. This is what we believe one of the most deep connection between *Morse homology* and *Floer cohomology*.

2.3 Differential

While in the previous section we have pointed out that *Floer cohomology* takes its origin from *Morse homology* and their differentials share a lot similar character, then we make this precise. As in *Morse homology*, the grading in *Floer cohomology* is also defined by counting number of curves, which is pseudo-holomorphic w.r.t. the almost complex structure we've chosen before. To be specific, for arbitrary intersection point p of Lagrangian L_0 and L_1, the Floer differential of p, ∂p, is a linear combination of the intersection points of L_0 and L_1:

$$\partial p = \sum_{q \in L_0 \cap L_1} \#\mathscr{M}(p, q) q, \tag{6}$$

where the summation is over all the intersection points satisfying some particular conditions we'd state below. Given a homotopy class $u \in \pi_2(M, L_0 \cup L_1)$, two intersection points $p, q \in L_0 \cap \mathrm{L}_1$, and an almost complex structure J on TM, we denote the moduli space of pseudo-holomorphic curves ϕ that satisfies the condition in equation 3 as $\widehat{\mathscr{M}}(p, q, [u], J)$, and we use $\mathscr{M}(p, q, [u], J)$ to denote the quotient space of $\widehat{\mathscr{M}}(p, q, [u], J)$ module the equivalence given by the $\mathbb{R}$-action on it:

$$\phi \sim \tilde{\phi} \qquad \exists a \in \mathbb{R}, \quad \phi(s, t) = \tilde{\phi}(s + a, t).$$

Since we have to count the number of a group of pseudo-holomorphic curves that satisfied the Cauchy-Riemann equation, which is an enumerative problem, first we have to consider whether the moduli problem, thus the moduli space is well-behaved and whether the number $\mathscr{M}(p, q)$ makes sense.

The first question, namely, the compactness of moduli space and transversality, as well as the independence on the choice of the almost complex structure J, will not be covered in this paper, because of their heaviness and lack of beautiful and marvelous inspiration, and curious readers can refer to the papers listed in the behind.

For the most essential problem to make sense of the number of this counting problem, we require the moduli space $\mathscr{M}(p, q, [u], J)$ of the pseudo-holomorphic curves to be discrete so that we can directly obtain the $\mathscr{M}(p, q)$ by counting. Also, remember that the $\mathbb{R}$-action on the moduli space $\widehat{\mathscr{M}}(p, q, [u], J)$, which indicates that this space should of one-dimension. Thus, the q appears in the summation should run around all the intersection points that have the moduli space of one-dimension, and we next try to determine the dimension of such a moduli space.

Briefly, there are two ways to determine the dimension of such a moduli space, an analytical approach and a topological approach, in this section we'll focus on the analysis approach via Fredholm operator and delay the discussion of the topological approach to the next section because of the connection between it and the definition of grading.

2.3.1 An Analytical Approach via Fredholm Operator

Since the pseudo-holomorphic curves satisfy the Cauchy-Riemann equation, we can view it as a problem concerned with the solution of an operator on the vector bundle of function on the manifold. To compute the dimension of a manifold, it suffices to compute the dimension of its tangent space at a given point, which is a much easier problem. Also, the boundary value problem concerned with 3 is a Fredholm problem, i.e. the linearization $D_{\bar{\partial}_J}u$ of at a solution u is a Fredholm operator. Specifically, Specifically, $D_{\bar{\partial}_J}u$ is a $\bar{\partial}$-type first-order differential operator, whose domain is a suitable space of sections of the pullback bundle u^*TM (with Lagrangian boundary conditions), for example $W^{1,p}(\mathbb{R}\times[0,1],\mathbb{R}\times\{0,1\};u^*TM,u^*_{|t=0}TL_0,u^*_{|t=1}TL_1)$. By the elementary theory of index of operators, we conclude that the desired dimension of the solution space equals the Fredholm index $\operatorname{ind}([u]) := \operatorname{ind}_{\mathbb{R}}(D_{\bar{\partial}_J}u) = \dim\operatorname{Ker} D_{\bar{\partial}_J}u - \dim\operatorname{Coker} D_{\bar{\partial}_J}u$. Thus in the summation, it suffices to consider the intersection point q and the homotopy class of $[u]\in\pi_2(M, L_0\cup L_1)$such that the index $\operatorname{ind}(u)=1$. Notice the index $\operatorname{ind}(u)$ can be computed in terms of an invariant of the class $[u]$ called the Maslov index, which is the topological approach we've mentioned before that will discuss below.

In all, summarize all the thing we've derived right now, we can give the following definition:

Definition 1. *The Floer differential $\partial: CF(L_0,L_1)\to CF(L_0,L_1)$ is the $\mathbb{K}$-linear map defined by:*

$$\partial(p) = \sum_{\substack{q\in\mathcal{X}(L_0,L_1)\\ [u]:ind([u])=1}} \mathscr{M}(p,q;[u],J)T^{\omega([u])}q \tag{7}$$

where $\#\mathscr{M}(p,q;[u],J)\in\mathbb{Z}(or\ \mathbb{Z}_2)$ is the signed (or unsigned) count of points in the moduli space of pseudo-holomorphic strips connecting p to q in the class $[u]$ satisfies 3, and $\omega([u]) = \int_{\mathbb{R}} u^*\omega$ is the symplectic area of those strips.

Remark 1. *The issue of sign concerned with this linear combination is pretty tricky here. It's closely related to the orientability of the moduli space. And we don't further touch this issue with the guarantee that the orientability is easy to handle in the example considered by us.*

To guarantee the vanish of twice boundary, we have the following result due to Floer:

Theorem 2. *Assume that $[\omega]\cdot\pi_2(M,L_0)=0$ and $[\omega]\cdot\pi_2(M,L_1)=0$. Moreover, when $char(\mathbb{K})\neq\mathbb{Z}_2$ assume that L_0, L_1 are oriented and equipped with spin structures. Then the Floer differential ∂ is well-defined,a nd the Floer cohomology $HF(L_0,L_1)=H^*(CF(L_0,L_1),\partial)$ is, up to isomorphism, independent of the chosen almost-complex structure J and invariant under Hamiltonian isotopies of L_0 or L_1.*

Remark 2. *Note that in the above theorem the Floer cohomology is not associated with a grading. The problem of grading is concerned with the Maslov index which is the index of some loop in Lagrangian Grassmannian. Such an index gives grading to each intersection point and therefore provide a grading for the whole complex.*

2.4 Grading

To give a grading on the Floer complex, we have to require further condition on Lagrangian and the symplectic manifold.

For symplectic manifold M: its first Chern class must be 2-torsion: $2c_1(TM)=0$. This guarantees the existence of a fiber-wise universal covering space $\widetilde{LGr}(TM)$.

For Lagrangian L, we require that the Maslov class $\mu_L\in Hom(\pi_1(L),\mathbb{Z})$ vanish. For arbitrary loop in L, it corresponds to a loop in the space $\mathscr{L}(n)$, namely, the Grassmannian of the Lagrangian subspace in $\mathbb{R}^{2n}$, then since $\mathscr{L}(n)\simeq U(n)/O(n)$, we have $\pi_1(\mathscr{L}(n))\simeq\pi_1(U(n)/O(n))\simeq\mathbb{Z}$ and the Maslow class follows by this composition.

Such requirements seem abstract, but we illustrate it using the example in hand, namely $T*S^1\simeq S^1\times\mathbb{R}$, and the condition for Lagrangian to satisfy coincides with that we given in the first subsection, it bounds region with vanishing symplectic area.

Remark 3. *It'll be interesting to find whether these two conditions are talking about the same thing.*

For symplectic manifold and its Lagrangian submanifold satisfy this condition, then we can lifting the Maslov map class from S^1 to $\mathbb{R}$, and this lift corresponds to give a grading on this Lagrangian, that's, in 2-dim, instead of a point on S^1 which indicates the direction of the Lagrangian tangent subspace, we require it to be chosen in $\mathbb{R}$.

After having chosen a grading on Lagrangian L_0 and L_1, then we can define a grading on the Floer chain complex $CF(L_0, L_1)$, that's, for each intersection point, we can canonically associate with it a degree in $\mathbb{Z}$.

The following is an example that illustrates the procedure we discussed previously.

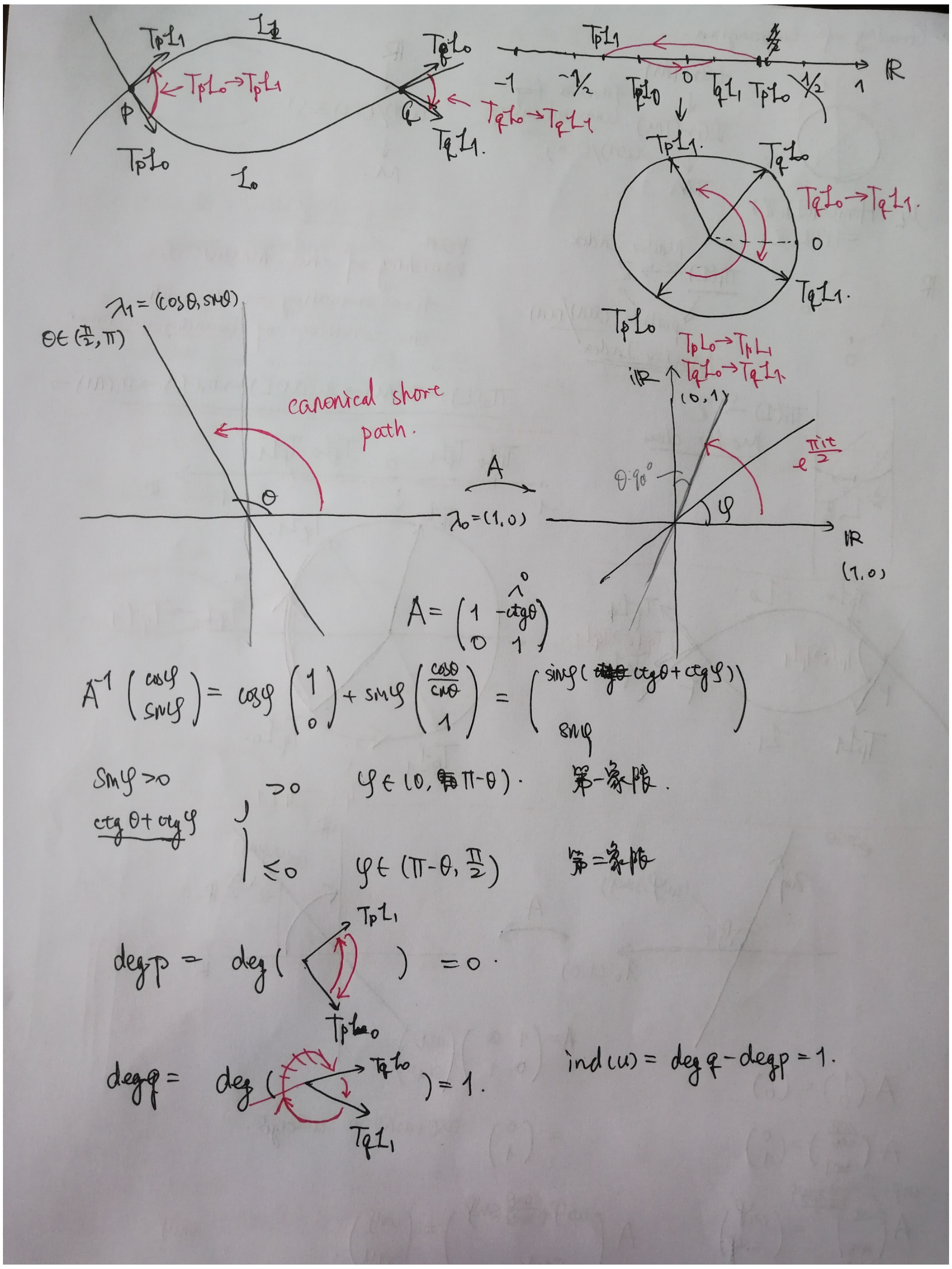

Figure 2: Maslov-index

2.4.1 A Toy Example for Calculating Floer Homology

In this subsection, we'll give a direct calculation on the *Floer cohomology* we've defined previously. The ground manifold is still S^1 with $T^*S^1 \simeq S^1 \times \mathbb{R}$. Now suppose Lagrangian L is the zero section and we have a hamiltonian $H = \cos\theta$(θ is the global coordinate on $T^*S^1 \simeq S^1 \times \mathbb{R}$), whose induced flow is given by:

$$X_H = -\sin\theta \frac{\partial}{\partial r}. \tag{8}$$

and denote the hamiltonian isotopy as ϕ, we aim to prove that $HF(L, \phi(L)) \simeq H^*(L; \mathbb{Z})$.

As shown in the next figure, the Lagrangian L and $\phi(L)$ intersects at two points, namely p and q. Because the Maslov classes vanish for these two Lagrangian, we can provide a lifting thus a grading for this two Lagrangian, the grading of Lagrangian L is constant lifting, and the grading for $\phi(L)$ is just like:

$$\gamma(t) = \begin{cases} t & t \in [0, \frac{1}{4}), \\ \frac{1}{2} - t & t \in [\frac{1}{4}, \frac{3}{4}), \\ t - 1 & t \in [\frac{3}{4}, 1]. \end{cases}$$

Then the grading of p, q in $HF(L_0, L_1)$ is easily calculated as 0 and 1. Thus the Floer complex has two $\mathbb{Z}$ summand in degree $0, 1$. Notice that this complex is isomorphic to the one concerned with that of S^1. Next, we compute the Floer differential of p, q to show these two are isomorphic. Recall that the coefficient appears in the grading is related to counting the number of pseudo-holomorphic curves. Since p and q are symmetric, it suffices to calculate the coefficient of q in the differential of p. We have to count the pseudo-holomorphic curves which are bounded by two Lagrangian L_0, L_1 with two ends converge to p, q respectively. It's easy to observe that there are two such strips satisfy the requirement, the one on the front and the one on the behind. Calculating the index of p, q demonstrates that these two strips correspond to the converse sign. Consequently, the total coefficient in the differential is 0. And we conclude that the Floer differential at each degree is given by 0

$$\cdots \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \cdots.$$

The Floer cohomology is thus concentrating in degree 0 and 1, which coincides with that of $\mathbb{S}^1$.

2.5 Product Structure

Using the same philosophy to define the differential of *Floer cohomology*, a beautiful product structure can be established on the it, which will be a main construction block for the later discussed *Fukaya Category* and now we provide a brief illustration.

For three Lagrangian L_0, L_1, L_2 and intersection points $q \in L_0 \cap \mathrm{L}_2, p_1 \in L_0 \cap \mathrm{L}_1, p_2 \in L_1 \cap \mathrm{L}_2$, we define the coefficient of q in the product of p_1 and p_2 as the number of moduli space of pseudo-holomorphic maps

Definition 2. *The product in Floer cohomology: $CF(L_0, L_1) \otimes CF(L_1, L_2) \to CF(L_0, L_2)$ is the $\mathbb{K}$-linear map defined by:*

$$p_1 \cdot p_2 = \sum_{\substack{q \in \mathcal{X}(L_0, L_2) \\ [u]: ind([u]) = 0}} \mathscr{M}(p_1, p_2, q; [u], J) T^{\omega([u])} q \tag{9}$$

where $p_1 \in L_0 \cap L_1, p_2 \in L_1 \cap L_2$, $\#\mathscr{M}(p_1, p_2, q; [u], J) \in \mathbb{Z}$(or $\mathbb{Z}_2$) is the signed (or unsigned) count of points in the moduli space of pseudo-holomorphic maps mapping from the disk with three punctured point to the domain bounded by L_0, L_1, L_2 and three punctured points map to p_1, p_2, q in the class $[u]$ again satisfies the Riemann-Cauchy equation similar to 3, and $\omega([u]) = \int_{\mathbb{R}} u^*\omega$ is the symplectic area of those strips.

The product structure is compatible with the Floer differential in the sense that they satisfy Leibniz rule together:

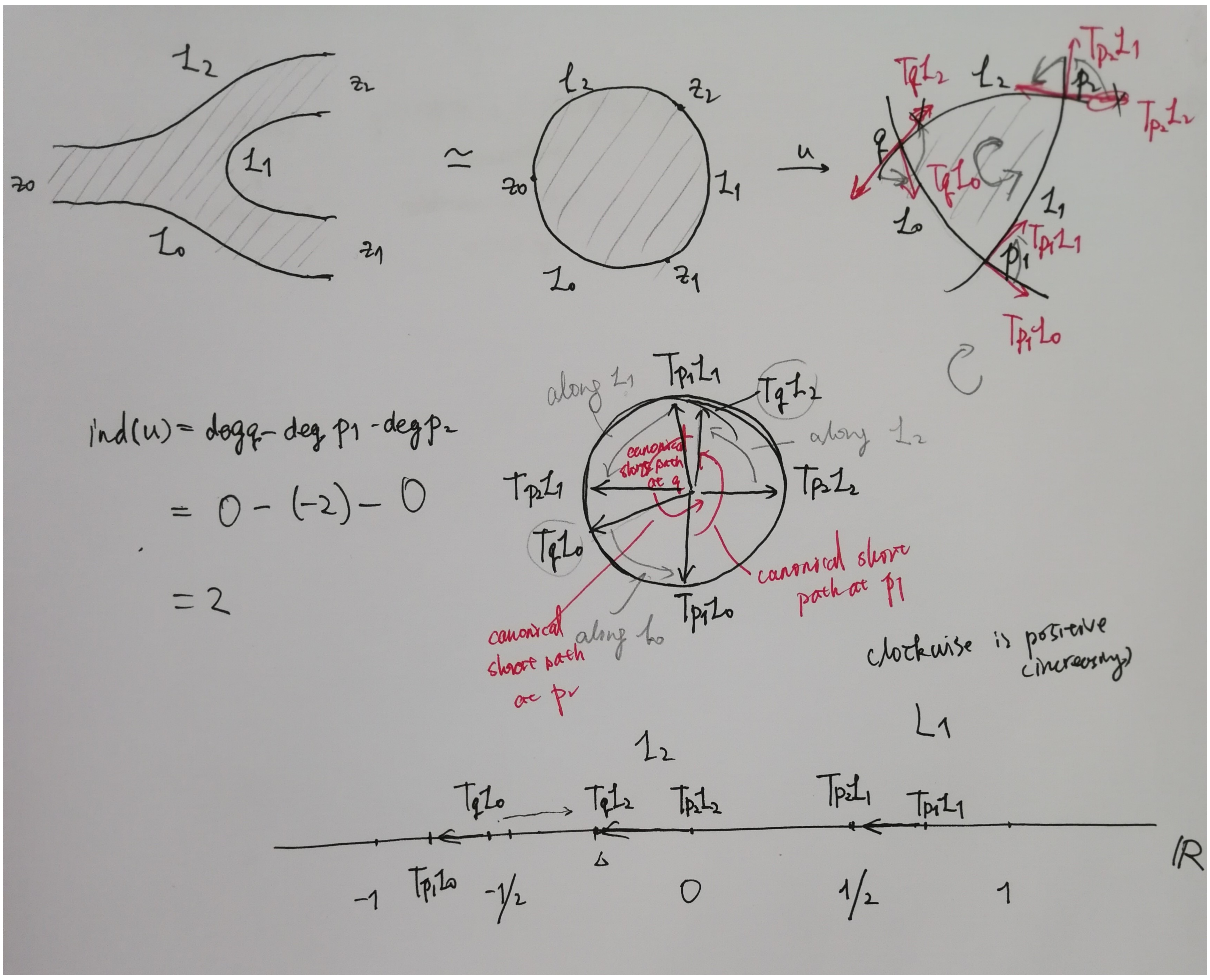

Figure 3: Product Structure

Definition 3. *The Floer differential obeys Leibniz law w.r.t. the product structure 2 defined above:*

$$\partial(p_1 \cdot p_2) = \partial p_1 \cdot p_2 + p_1 \cdot \partial p_2 \tag{10}$$

2.5.1 Higher Product

Completely similar to the definition in product case, we can define the higher product structure associated with the *Floer cohomology* and we introduce symbol μ^k to represent all of them, we stated it directly here.

Definition 4. *The product in Floer cohomology:* $CF(L_0, L_1) \otimes CF(L_1, L_2) \otimes \cdots CF(L_{k-1}, L_k) \to CF(L_0, L_k)$ *is the* $\mathbb{K}$*-linear map defined by:*

$$\mu^k(p_1, p_2, \cdots, p_k) = \sum_{\substack{q \in \mathcal{X}(L_0, L_k) \\ [u]: ind([u]) = 2-k}} \mathscr{M}(p_1, p_2, \cdots, p_k, q; [u], J) T^{\omega([u])} q \tag{11}$$

where $p_1 \in L_0 \cap L_1, p_2 \in L_1 \cap L_2$*,* $\#\mathscr{M}(p_1, p_2, q; [u], J) \in \mathbb{Z}(or\ \mathbb{Z}_2)$ *is the signed (or unsigned) count of points in the moduli space of pseudo-holomorphic maps mapping from the disk with three punctured point to the domain bounded by* L_0, L_1, L_2 *and three punctured points map to* p_1, p_2, q *in the class* $[u]$ *again satisfies the Riemann-Cauchy equation similar to 3, and* $\omega([u]) = \int_{\mathbb{R}} u^*\omega$ *is the symplectic area of those strips.*

The most important relationship between all of this product structure is the A_∞-relations, which can be viewed as the high dimension analogue of the Leibniz law 3 and Floer differential 1:

Definition 5. *The product μ^k defined above satisfy the A_∞-relations in the following sense:*

$$\sum_{m,n}(-1)^{|a|-n}\mu_{\mathscr{A}}^{d-m+1}(a_d,\cdot\cdot,a_{n+m+1},\mu_{\mathscr{A}}^{m}(a_{n+m},\cdot\cdot,a_{n+1}),a_n,\cdot\cdot,a_1)=0$$

Remark 4. *The mysterious vain of the above formula doesn't show is power until we formulate the notion of A_∞-category. Therefore, we don't explain here carefully this formula.*

3 Fukaya Category

After the preliminary discussion, we can finally give a relatively strict definition of the *Fukaya Category* though we still have to omit some detail about the perturbation issue about transitivity and compactness.

Definition 6. *Let (M,ω) be a symplectic manifold with $2c_1(TM)=0$. The objects of the (compact) Fukaya category $\mathscr{F}(M,\omega)$ are compact closed, oriented, spin Lagrangian submanifolds $L\subset M$ such that $[\omega]\cdot\pi_2(M,L)=0$ and with vanishing Maslov class $\mu_L=0\in H_1(L,\mathbb{Z})$, together with extra data, namely the choice of a spin structure and a graded lift of L. (We will usually omit those from the notation and simply denote the object by L.) For every pair of objects (L,L') (not necessarily distinct), we choose perturbation data $H(L,L')\in C^\infty([0,1]\times M,\mathbb{R})$ and $J(L,L')\in C^\infty([0,1],J(M,\omega))$; and for all tuples of objects $(L_0,...,L_k)$ and all moduli spaces of discs, we choose consistent perturbation data (H,J) compatible with the choices made for the pairs of objects (L_i,L_j), so as to achieve transversality for all moduli spaces of perturbed J-holomorphic discs. (See [15], ch 9 for the existence of such perturbation data.) Given this, we set $hom(L,L')=CF(L,L';H(L,L'),J(L,L'))$; and the differential μ^1, composition μ^2, and higher operations μ^k are given by counts of perturbed pseudo-holomorphic discs as in Definition 1 2 4. Thie makes $\mathscr{F}(M,\omega)$ a Λ-linear, $\mathbb{Z}$-graded, non-unital (but cohomologically unital [15]) A_{infty}-category.*

Remark 5. *We comment here on the motivation of Fukaya category, one is from the Floer cohomology, which is the essence and structure of Fukaya Category, the other is from the point of view from Gromov-Witten invariant, Fukaya Category can be viewed as an improvement of the Gromov-Witten invariant theory. In [14], it's pointed out that because of the assumption we take for Lagrangian in the manifold(it doesn't bound holomorphic disc with area), the Gromov-Witten invariant will be trivial in this case, thus we shift our attention to the case that we remove several points on the boundary of the disc and then do the counting stuff, which is exactly the philosophy for building Fukaya Category.*

3.1 On Homological Theory of Fukaya Category

In this section we aim to find some interesting homological aspect of the *Fukaya Category*, namely, the generators, and also some mapping cone in the *Fukaya Category*, most of this result is developped and illustrated in symplectic geometry which is proved by classical symplectic geometry, and viewing them from the perspective of homological theory truly shade light on this delicate geometric structure.

We first talk about the generator of the *Fukaya Category*

Now we move onto the topic of mapping cones in *Fukaya Category*, the two examples we shall discuss is the Dehn twist and Lagrangian connected sum.

3.2 A Toy Example for Calculating Fukaya Category

In this subsection we provide a direct calculation of the fukaya category on $T^*S^1=S^1\times\mathbb{R}$. This calculation correspond so-called wrapping fukaya category, which illustrate the point that the category is formed by wrpping the Lagrangian and then compute *Floer cohomology* between them.

First we consider a cotangent fibre L on T^*S^1, that's $L=T_q^*S^1$. Notice that although cotangent fibre is the most trivial and basic object in the fukaya category, Abouzaid has shown in his beautiful paper that the wrapped fukaya category is generated by those cotangent fibre [1].

Here we'd like to quote some comments from that of

The wrapped Floer cohomology, which we denoted as $HW(L,L)$, is associated with a given hamiltonian flow ϕ, this ϕ is required to satisfy some condition, which corresponds to the big perturbation around the infinity.

3.3 Wrapped Fukaya Category

In some case, not only do we consider compact Lagrangians, but also we study non-compact Lagrangians which have controllable behaviour in infinity, such consideration becomes significantly important when we are dealing with the *Fukaya Category* of cotangent bundle—since canonical Lagrangians like cotangent fibres are definitely non-compact but can be easily controlled at infinity, and also gives birth to other important *Wrapped Fukaya Category* towards systematically understanding the mirror symmetry. Thus in here we spend a whole subsection to explore some beautiful idea that's concerned with *Wrapped Fukaya Category*. Following two main philosophy that guiding the study of *Wrapped Fukaya Category*, we mainly discuss the *Wrapped Fukaya Category* introduced by Nadler, Zaslow and Abouzaid, Seidel.

3.3.1 Wrapped Fukaya Category and Based Loops

The work by Abouzaid [1] [2]is mainly concerned with the geometric and algebraic relationship between the cotangent fibre and base manifold, which is bridging by the whole cotangent bundle. Specificly, he establishs an A_∞ isomorphism between the *Wrapped Fukaya Category* associated with a cotangent fibre and the chain on the loop space of the base manifold. We stated several main results here and then use some geometric intuition to discuss this and give some example computation on it.

Theorem 3. *If Q is a closed smooth manifold, there exists an A_∞ equivalence:*

$$CW_b^*(T_q^*Q) \to C_{-*}(\Omega_q Q) \tag{12}$$

between the homology of the space of loops on Q based at q and the Floer cohomology of the cotangent fibre at q taken as an object of the Wrapped Fukaya Category of T^*Q with background class $b \in H^*(T^*Q, \mathbb{Z}_2)$ given by the pullback of $\omega_2(Q) \in H^*(Q, \mathbb{Z}_2)$

Remark 6. *We give here an informal remark that such result between the cotangent fibre and the based manifold is bridged by the ambient space, namely, the cotangent bundle T^*Q. But we have to say that this approach doesn't have much connection to the Morse cohomology, which is the origin of the Floer cohomology, which forms sharp contrast to the approach by Nadler and Zaslow of Inifinitesimal Wrapped Fukaya Category, whose guiding philosophy is the common origin of Morse cohomology and we'll carefully discuss this point in the next subsection. For now, we comment that the connection in this approach is just a geometric intuition of the holomorphic strips have boundary as loops in based manifold.*

3.3.2 Infinitesimal Wrapped Fukaya Category

In this section we briefly talked about the introduction of *Inifinitesimal Wrapped Fukaya Category* introduced by Nadler and Zaslow to formulate a Mirror Symmetry result on the cotangent bundle. Though both the goal of the approach in this subsection and the previous subsection are to study the property of cotangent bundle, we'll see below that these two approached are intrinsically different.

4 Preliminary on A_∞-Category

First we give the definition of the A_∞-category, which is an abstraction of the Fukaya category. follow the philosophy that we want to illustrate the underlying reason of construct such a complicated category structure, and our source diverse in a great range of classical mathmatics like *Gromov-Witten invariant*.

4.1 A_∞-Category

Fix a coefficient field $\mathbb{K}$, all of our category is assumed to be linear over $\mathbb{K}$ and small.

Definition 7. *An non-unital A_∞-category $\mathscr{A}$ is given by a set of objects $Ob\mathscr{A}$, $\mathbb{Z}$-graded vector space $hom_{\mathscr{A}}(X_0, X_1)$, and multilinear composition map:*

$$\mu^d_{\mathscr{A}} : hom_{\mathscr{A}}(X_0, X_1) \otimes hom_{\mathscr{A}}(X_1, X_2) \cdots hom_{\mathscr{A}}(X_{d-1}, X_d) \to hom_{\mathscr{A}}(X_0, X_d)[2-d] \tag{13}$$

where the $[k]$ refers to shift the grading of the vector space down by $k \in \mathbb{Z}$. And we have the following identity between those composition maps called (quadratic) A_∞-associativity equations:

$$\sum_{m,n} (-1)^{|a|-n} \mu^{d-m+1}_{\mathscr{A}}(a_d, \cdot\cdot, a_{n+m+1}, \mu^m_{\mathscr{A}}(a_{n+m}, \cdot\cdot, a_{n+1}), a_n, \cdot\cdot, a_1) = 0 \tag{14}$$

where the summation is over all possible $m \leq d$ and $n \leq d-m$ and $|a| = \sum_{i=1}^n |a_i|$ and $|a_i|$ refers to the degree of a_i in $hom_{\mathscr{A}}(X_i, X_{i+1})$.

Remark 7. *the non-unital means that not necessary unital instead of not unital. Notice that $\mu^d_{\mathscr{A}}$ is a $\mathbb{Z}$-graded morphism between $\mathbb{Z}$-graded vector space, and is equivalent to graded morphism :*

$$\mu^d_{\mathscr{A}} : hom_{\mathscr{A}}(X_0, X_1)[i_0] \otimes hom_{\mathscr{A}}(X_1, X_2)[i_1] \cdots hom_{\mathscr{A}}(X_{d-1}, X_d)[i_{d-1}] \to hom_{\mathscr{A}}(X_0, X_d)[\sum_{j=0}^{d-1} i_j + 2 - d] \tag{15}$$

thus in the A_∞-associativity equations we have morphism between:

$$\begin{aligned} &hom_{\mathscr{A}}(X_0, X_1) \otimes \cdots hom_{\mathscr{A}}(X_{n-1}, X_n) \otimes hom_{\mathscr{A}}(X_n, X_{n+m})[2-m] \\ &\otimes hom_{\mathscr{A}}(X_{n+m}, X_{n+m+1}) \cdot\cdot hom_{\mathscr{A}}(X_{d-1}, X_d) \to \\ &hom_{\mathscr{A}}(X_0, X_d)[2-m+2-d+m-1] = hom_{\mathscr{A}}(X_0, X_d)[3-d] \end{aligned} \tag{16}$$

Also, by direct computation $(d = 1, m = 1, n = 0)$ we have :

$$\mu^1_{\mathscr{A}}(\mu^1_{\mathscr{A}}(a)) = 0 \tag{17}$$

which turns the $\mathbb{Z}$-graded vector space $(hom_{\mathscr{A}}(X_0, X_1), \mu^1_{\mathscr{A}})$ into a chain complex. Thus we can associate with $\mathscr{A}$ its cohomological category $H(\mathscr{A})$ with the same objects as $\mathscr{A}$ and morphism given by $H(hom_{\mathscr{A}}(X_0, X_1), \mu^1_{\mathscr{A}})$, and the composition is given by:

$$[a_2] \cdot [a_1] = (-1)^{|a_1|}[\mu^2_{\mathscr{A}}(a_2, a_1)] \tag{18}$$

the well-defineness is given by the A_∞-associativity equations 14 for $\mu^2_{\mathscr{A}}$:

$$\mu^2_{\mathscr{A}}(a_2, \mu^1_{\mathscr{A}}(a_1)) + (-1)^{|a_1|-1}\mu^2_{\mathscr{A}}(\mu^1_{\mathscr{A}}(a_2), a_1) + \mu^1_{\mathscr{A}}(\mu^2_{\mathscr{A}}(a_2, a_1)) = 0 \tag{19}$$

which has some similarity with the formula of cup product in the classical cohomology theory. The above identity 19 can also be viewed as the Leibniz law for $\mu^1_{\mathscr{A}}$ w.r.t. $\mu^2_{\mathscr{A}}$. And the associativity of the composition 18 is given by the A_∞-associativity equations 14 for $\mu^3_{\mathscr{A}}$ which we'll not write here for simplicity. Thus it can be seen that the cohomological category $H(\mathscr{A})$ of the A_∞ category $\mathscr{A}$ is indeed a usual category.

Remark 8. *The A_∞ relation origins in the classical Floer homology theory which is generalized by Fukaya, notice the coincidence between the equation 14 and the shifting term $2-d$ origins from the counting of Maslov degree in the Floer homology. Another direction of connection with the classical theory is the well-known associativity law appears in quantum cohomology,*

4.2 A_∞-Functor

Then we have to associate with those A_∞-category functor between them. If we view A_∞-category as set of Lagrangians on symplectic manifold, then functor between them may origins from a morphism between two manifolds that preserve the symplectic structure. Thus it's reasonable to think that those functors have to be compatible with the A_∞ structure on the category, which we'll state below.

Definition 8. *A non-unital A_∞-functor between two non-unital A_∞-categories gives a map $\mathscr{F}\colon Ob\mathscr{A} \to Ob\mathscr{B}$ and multilinear maps:*

$$\mathscr{F}^d : hom_{\mathscr{A}}(X_0, X_1) \otimes hom_{\mathscr{A}}(X_1, X_2) \cdots hom_{\mathscr{A}}(X_{d-1}, X_d) \to hom_{\mathscr{B}}(\mathscr{F}X_0, \mathscr{F}X_d)[1-d] \tag{20}$$

which is compatible with the A_∞ structure on two categories in the following sense:

$$\begin{aligned} &\sum_r \sum_{s_1,\ldots,s_r} \mu^r_{\mathscr{B}}(\mathscr{F}^{s_r}(a_d,\ldots,a_{d-s_r+1}),\ldots,\mathscr{F}^{s_1}(a_{s_1},\ldots,a_1)) \\ &= \sum_{m,n} (-1)^{|a|-n} \mathscr{F}^{d-m+1}(a_d, \cdot\cdot, a_{n+m+1}, \mu^m_{\mathscr{A}}(a_{n+m}, \cdot\cdot, a_{n+1}), a_n, \cdot\cdot, a_1) \end{aligned} \tag{21}$$

where the LHS sum is over all $r \geq 1$ and $\sum s_i = r$, note that this identity can be viewed as some kind of commutativity between the morphism $\mathscr{F}$ and two composition maps $\mu_{\mathscr{A}}, \mu_{\mathscr{B}}$.

Remark 9. *We call $\mathscr{F}$ is a quasi-isomorphism(cohmologically full and faithful) if $H(\mathscr{F})$ is.*

$$\begin{CD} A @>{\varphi}>> B \\ @V{\rho_f}VV @VV{\rho_g}V \\ A_f @>{\phi_f}>> B_g \end{CD}$$

4.3 A_∞-Modules

The philosophy to first characterize the desired object in terms of its putative representing functor gives the first intuition to study the A_∞-modules, which is special class of A_∞ functors, the counterpart of the representable functor in the classical category theory. The point here is that the notion of Set Category $\mathscr{S}et$ in classical category theory has to be generalized to the context of A_∞ category, which we take as the Ch, the dg category of complex of $\mathbb{K}$-vector spaces, where the μ^1_{Ch} as the composition with the boundary in the complex and μ^2_{Ch} as ordinary composition of chain map while the higher compositions equal 0.

Definition 9. *The A_∞-module on an A_∞ category $\mathscr{A}$ is an A_∞ functor $\mathscr{M}\colon \mathscr{A} \to Ch$ with the associated composition map:*

$$\mu^d_{\mathscr{M}} : \mathscr{M}(X_{d-1}) \otimes hom_{\mathscr{A}}(X_{d-1}, X_{d-2}) \otimes hom_{\mathscr{A}}(X_{d-2}, X_{d-3}) \cdots \otimes hom_{\mathscr{A}}(X_1, X_0) \to \mathscr{M}(X_0)[2-d] \tag{22}$$

Note that we omit the compatible condition 21 for an A_∞ functor to satisfy. The set of non-unital A_∞-modules is denoted by nu-mod($\mathscr{A}$) =nu-fun($\mathscr{A}^{op}$, Ch).

Remark 10. *Comparing the compostion map for A_∞-module 22 and the composition map in the ordinary A_∞-category 13, the similarity indicates there is a canonical way to define a functor from the A_∞-category $\mathscr{A}$ to the functor category nu-mod($\mathscr{A}$):*

$$\begin{aligned} \ell_{\mathscr{A}} &: \mathscr{A} \to nu\text{-}mod(\mathscr{A}) \\ Y &\mapsto \mathscr{Y}(X) = hom_{\mathscr{A}}(X, Y) \end{aligned} \tag{23}$$

where the composition coincides with the A_∞ composition in $\mathscr{A}$. And thus we get the Yoneda Embedding in the context of A_∞-category.

$$\begin{CD} A @>{\varphi}>> B \\ @V{\rho_f}VV @VV{\rho_g}V \\ A_f @>{\phi_f}>> B_g \end{CD}$$

4.4 Triangulation

4.4.1 An Abstract Approach

Similar to the triangulation process of $D(\mathscr{A})$, the derived category of an abelian category $\mathscr{A}$, we'd like to now formulate the triangulation in the context of A_∞-category, which help us to utilize the power of language in derived category. The idea to establish triangulated structure is again working in the category of representation functor, i.e. establish triangulated structure in the nu-mod($\mathscr{A}$), then pull back this to get a triangulated structure in $\mathscr{A}$ via the *Yoneda-Embedding* $\ell_{\mathscr{A}}$. Since the definition of A_∞-category provides a natural triangulated structure on the morphism set while the objects in the nu-mod($\mathscr{A}$) are exactly those morphism space, such a observation make such definition possible.

In first, we have to establish a reverse correspondence between nu-mod($\mathscr{A}$) and $\mathscr{A}$ which can be view as a pseudo-inverse of the *Yoneda-Embedding*. We call that an object $Y \in \mathscr{A}$ quasi-represents the functor $\tilde{\mathscr{Y}} \in$ nu-mod($\mathscr{A}$) if there exists isomorphism in the category H^0(nu-mod($\mathscr{A}$)):

$$[t] : \mathscr{Y} \to \tilde{\mathscr{Y}} \tag{24}$$

And use the natural triangulated structure(chain complex structure) in the nu-mod($\mathscr{A}$), we can first push the object in $\mathscr{A}$ forward by the *Yoneda-Embedding* to nu-mod($\mathscr{A}$), and finish the operation in this context then pull it back along the quasi-representation:

$$\begin{aligned}
\textbf{Direct Sum}: &\quad \mathscr{Y}_0 \oplus \mathscr{Y}_1 \\
\textbf{Tensor Product}: &\quad \mathscr{Y}_0 \otimes Z, \qquad Z \in Ch \\
\textbf{Shift}: &\quad \mathscr{Y}_0[1](X) = hom_{\mathscr{A}}(X, Y)[1] \\
\textbf{Cone}: &\quad \mathscr{Y}_0 \oplus \mathscr{Y}_1[1]
\end{aligned} \tag{25}$$

Thus, by the compatibility between the A_∞-functor and the exact triangle, we have the following beautiful result which indicate the intrinsic connection between A_∞-category and triangulated category.

Proposition 1. *Suppose that $\mathscr{A}, \mathscr{B}$ are two triangulated A_∞-category and $\mathscr{F}$ an A_∞-functor between them, then $\mathscr{F}$ maps exact triangles to exact triangles.*

4.4.2 Twisted Complexes Approach

While the abstract approach developped in the last subsection has the virtue to be universal and several important properties followed clearly during the establishment, sometimes we still prefer a more solid approach to view the triangulated structure of an A_∞-category, thus in below we'll provide another formulation via twisted complexed.

We consider the situation to find the triangulated envelope of a given A_∞-category $\mathscr{A}$, which is defined as a pair $(\tilde{\mathscr{A}}, \mathscr{F})$, s.t. $\tilde{\mathscr{A}}$ a triangulated A_∞-category and the A_∞-functor $\mathscr{F}$:

$$\mathscr{F} : \mathscr{A} \to \tilde{\mathscr{A}} \tag{26}$$

is cohomologically full and faithful, with the image of $\mathscr{F}$ the generators of $\mathscr{B}$. Notice that the triangulated envelope of arbitrary A_∞-category always exists and is up to isomorphism.

We briefly discuss the construction of the twisted complexes approach to obtain the triangulated envelope of a given A_∞-category, the key idea is that we use some explicit construction, that's direct sum, tensor product, shift, mapping cone, to guarantee the existence of several key structure in the triangulated category. For the direct sum, we try to enlarge the category to make direct sum explicit defined, that is, we construct an A_∞-category $\Sigma\mathscr{A}$:

$$Ob\Sigma\mathscr{A} = \{\oplus_{i \in I} V^i \otimes X^i | X^i \in \mathscr{A}, V^i \in Ch\} \tag{27}$$

in [15]

$$\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
{\scriptstyle \rho_f}\downarrow & & \downarrow{\scriptstyle \rho_g} \\
A_f & \xrightarrow{\phi_f} & B_g
\end{array}$$

4.4.3 Dehn Twist and Lagrangian Connected Sum

In this subsection we provide some examples that illustrate the previous abstract construction in the A_∞-category. The interested reader can refer the reference [13].

Here is a result mentioned in the Seidel's survey paper [14] which indicates that to some extent, dehn twist and some distinguish basis can generate the whole A_∞-category. Formally, if we choose a distinguished basis of vanishing cycles $S_1, ..., S_m$ for the pencil, the product of their Dehn twists is almost the identity map. More precisely, taking into account the "grading" of the objects of the Fukaya category, one finds that:

$$\tau_{S_1}\tau_{S_2}\cdots\tau_{S_m}(L) = L[2] \tag{28}$$

Then by applying the exact triangle associated with the dehn twist, we get the following theorem:

Theorem 4. *$S_1, ..., S_m$ are split-generators for $D^\pi(F(M))$. This means that any object of $Tw^\pi(F(M))$ can be obtained from them, up to quasi-isomorphism, by repeatedly forming mapping cones and idempotent splittings.*

5 Homological Mirror Symmetry

5.1 A Toy Example

In this subsection we will give a direct manifestation of the homological mirror symmetry.

5.1.1 A Glimpse of the Toy Example

First is a completely naive one, where we still use our favourite S^1 model. We'll first provide some informal argument and try to extract the essence that is shown in this section. In this subcase, the goal of the HMS is to establish a category equivalence between the bounded derived category of the *Fukaya Category* associated with the cotangent bundle of S^1, i.e. $D^b(Fuk(T^*S^1))$, and the bounded derived category of constructible sheaves on the S^1, i.e. $D^b_c(S^1)$. Namely, we intuitively provide a pair of map between these category and then prove there inverse to each other, that their composition is identity in two categories and they induced isomorphism on the *hom*-set, which is A_∞-equivalence.

On the one side, the map from $D^b_c(S^1)$ to $D^b(Fuk(T^*S^1))$ is simply given by the micro-support functor SS introduced by Shapira and Kashiwara in their famous book [9]

$$\begin{aligned} SS : D^b_c(S^1) &\to D^b(Fuk(T^*S^1)) \\ \mathscr{F} &\mapsto SS(\mathscr{F}) \end{aligned} \tag{29}$$

it's well-known that micro-support of a sheaf is a coisotropic conic Lagrangian in the cotangent bundle.

On the other side, the inverse of this micro-support functor is sometimes called "quantization" with reason not known to myself. Following the definition of that in the work of Viterbo [17], it's given by associating to a Lagrangian L the sheaf of complex $\mathscr{F}$ on S^1 such that its stalk at each point $x \in S^1$ is $\mathscr{F}_x = CF(L, T^*_xS^1; \partial_x)$, where $CF(L, T^*_xS^1; \partial_x)$ is the Floer complex associated to the pair of Lagrangian L and cotangent fibre at x V_x with Floer differential ∂_x.

To show these two maps are in fact inverse to each other, we provide an intuitive idea here. First, we consider about the skyscrapper sheaf $\Bbbk_x$ standing at one point x, then we know that under the micro-support map we get the Lagrangian in $D^b(Fuk(T^*S^1))$ as the cotangent fibre beyond x:

$$SS(\Bbbk_x) = T^*_xS^1 \tag{30}$$

Now, consider the quantization map, we have that the image of $T^*_xS^1$ under the quantization map as a complex of sheaf $Q(T^*_xS^1) \in D^b_c(S^1)$ with stalk:

$$Q(T^*_xS^1)_y = CF(T^*_xS^1, T^*_yS^1; \partial_y) \tag{31}$$

since for two different point x, y, we know that their cotangent fibres don't intersect each other, thus we have:

$$Q(T^*_xS^1)_y = CF(T^*_xS^1, T^*_yS^1; \partial_y) = 0, \qquad x \neq y \tag{32}$$

and for $x = y$, we have to provide a perturbation on the cotangent fibre $T_x^* S^1$ so that they don't coincide identically anymore and elementary *Floer cohomology* theory show us:

$$Q(T_x^* S^1)_x = CF(T_x^* S^1, T_x^* S^1; \partial_y) = k \tag{33}$$

thus we've shown that there is isomorphism $Q(T_x^* S^1) \equiv \Bbbk_x$. And the mapping between the induced by considering a morphism between two objects in $D_c^b(S^1)$, $f : \mathscr{F}_1 \to \mathscr{F}_2$, taking the functor of micro-support SS, we

$$SS(f) : SS(\mathscr{F}_1) \to SS(\mathscr{F}_2) \tag{34}$$

Then we have to prove that there're also isomorphism between the morphism space. It's trivial that we have the following isomorphism:

$$hom_{D_c^b(S^1)}(\Bbbk_x, \Bbbk_y) = \varnothing. \tag{35}$$

6 Mirror Symmetry in Information Geometry

Recently, a discussion with Jun Zhang provide me with a new insight of mirror symmetry in information geometry, which concerned with the geometry of probability and statistics. In information geometry, both Kahler structure and almost complex structure can be defined, and the miraculous correspondence between such two geometric structure shows up in the context of statistic, which also provide us with an statistical illustration of the mirror symmetry.

Personally, mirror symmetry in information geometry is not so natural in my view because the complex and kahler structure they introduced into the statistical structure is not natural at the first glimpse. Namely, the most crucial thing they focus on is the Codazzi coupliing in the dual structure, which is an artificial compatible condition introduced to the geometry with statistic, complex, symplectic structure defined on it. I've talked to Prof. Zhang about the un-naturalness of the introduction of complex structure in the infromation geometry. Is there canonical or in some sense physics-origin complex structure we can have on the statistic manifold, his answer of introducing it via the method of divergence, which is prevalent in the field of information geometry doesn't convince me strongly. But here I still briefly summarize the work of JunZhang on the connection between information geometry and complex, symplectic structure on it, and the main reference is listed here [21] [5]

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8 Appendix

In this appendix we provide a brief introduction to the *Differential Graded Lie Algebra*(DGLA) approach to the deformation theory. As both DGLA and deformation theory interact a lot with the symplectic we discussed in this note, it's certainly an interesting topic to choose DGLA approach to deformation theory as an additional material.

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Personal statement of the mentor Prof. Lei Yang

I am interested in automorphic forms, number theory, algebraic geometry, representation theory, topology and mathematical physics. I am very interested in the interaction of different branches of mathematics and physics, I find it fascinating. Selected Papers: Lei Yang, Beijing (Peking) UniversityBeijing 100871, Peoples Republic of China, **Hessian polyhedra, invariant theory and Appell hypergeometric functions**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018. viii+308 pp. ISBN: 978-981-3209-47-3.